[the letter A] was used as a numeral denoting 500, or, with a dash over it, thus, $\bar{A}$, it stood for 500,000."

The question at once arises what ancients are meant. The number systems of the ancient Babylonians, the ancient Egyptians, the ancient Greeks, etc., do not reveal such a use of the letter A as a number symbol. From the fact that in medieval times the Romans sometimes used the horizontal bar above a number symbol to denote that the number represented by this symbol is to be multiplied by 1,000 leads one to suspect that the term "ancients" in this quotation probably refers to the Romans in the Middle Ages, but there is nothing in the article itself which would aid one in reaching this conclusion.

To teachers of mathematics the obsolete methods, viewpoints, and terms should be of peculiar interest since they involve elements which were once attractive, and were replaced by others which were still more attractive. The genesis of our modern methods and viewpoints may reveal possible further improvements, and a knowledge of the obsolete may enable us to assist more effectively in consigning to the obsolete those things which could now be replaced by the more useful.

A SUBSTITUTE FOR DUPIN'S INDICATRIX.

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1. **Dupin's Indicatrix.** Let the surface be given in the form $z = f(x, y)$ and let us take the origin at a non-singular point and take the tangent plane at the origin for the xy-plane. Then z can be expanded into an infinite series in x, y which will begin with the second-degree terms,

$$z = \tfrac{1}{2}(ax^2 + 2hxy + by^2) + \cdots, \tag{1}$$

where the terms omitted are of higher than the second order. One method of obtaining Dupin's indicatrix then is to write

$$ax^2 + 2hxy + by^2 = \pm 1. \tag{2}$$

If this conic is an ellipse then use the sign on the right, which will make it real and if it is an hyperbola use either sign. We note that in case this conic becomes a parabola it degenerates into two coincident lines and never takes the form of a general parabola. Points on the surface are then classified as elliptic, hyperbolic or parabolic according as the indicatrix is an ellipse, hyperbola or two coincident straight lines. The axes of the indicatrix correspond to the directions of the lines of curvature on the surface. The direction of the asymptotes of the indicatrix are the same as the direction of the asymptotic lines of the surface. Two directions on the surface are said to be conjugate if they coincide with conjugate diameters of the indicatrix. So we see that the indicatrix is quite intimately associated with the important directions on the surface.

It always seemed to me however that there was something arbitrary in the

choice of this conic because we take the second-degree terms and set them equal to a constant and be careful that the constant is so chosen that the conic is real. It would seem more reasonable to me to set the second-degree terms equal to zero. If this were done the two lines into which it factors would give the asymptotic directions and the bisectors of the angle between them would give the direction of the lines of curvature. Conjugate diameters do not depend on the equation of a conic but only on the directions of the asymptotes so that conjugate directions could be defined if we should take as directrix the two lines obtained by setting the second-degree terms equal to zero. The method used by de la Vallee-Poussin in his *Cours d'analyse* to obtain the Dupin indicatrix is not open to the above objection. He measures on each tangent line to the surface at O a length OT equal to the square root of the radius of normal curvature for that direction. The locus of T is the Dupin indicatrix.

If one wishes to study the theory of surfaces in higher dimensions he will find that the indicatrix does not easily generalize and that the sort of things which it shows about the surface have very slight connection with the sort of things which the indicatrix shows for surfaces in three dimensions. For these reasons I venture to offer a substitute for the indicatrix.

2. **Substitute for Dupin's Indicatrix.** In what follows we shall be more interested in curvature than the radius of curvature of a curve and shall use the term normal curvature at a point to denote the curvature of the section of the surface made by a plane which contains a tangent line and normal to the surface at that point. The normal curvature depends on the particular tangent line taken. The following discussion will be given only to show properties of the surface in the neighborhood of a point and for simplicity we will take the point under discussion for the origin and will use equation (1) for the equation of the surface. The substitute which I propose for the indicatrix is then obtained as follows: Measure on the normal to the surface at O (the axis of z) a distance OC equal to the normal curvature in any direction. The locus of C, as we vary the direction, is a section of the normal. The position of this segment with respect to the surface point (origin in this case) tells us as much about the character of the surface as does the indicatrix of Dupin. We shall see that this segment may, with advantage, be considered as a degenerate ellipse. This locus generalizes quite readily for higher dimensions, but in that case the locus is not, in general, a degenerate ellipse.

Let the axes of the conic (2) be taken for the x- and y-axes, that is let the x- and y-axes be tangent to the lines of curvature. The equation of the surface then takes the form

$$z = Ax^2 + By^2 + \cdots. \tag{3}$$

Any plane passing through the normal (z-axis) is

$$y = kx, \tag{4}$$

and any normal section is obtained by taking (3) and (4) simultaneously. We

will now find the curvature of such a normal section. The curvature of a twisted curve given in the form

$$x = f(t), \qquad y = g(t), \qquad z = h(t)$$

is

$$(5) \quad c = \sqrt{\frac{d^2x}{ds^2} + \frac{d^2y}{ds^2} + \frac{d^2z}{ds^2}} = \frac{[(g'h'' - h'g'')^2 + (g'f'' - f'g'')^2 + (h'f'' - f'h'')^2]^{1/2}}{[f'^2 + g'^2 + h'^2]^{3/2}},$$

where s denotes the arc length and primes denote derivatives with respect to t.[1] To apply formula (5) to the curve in hand we first substitute (4) in (3) and then the parametric equations of the curve become

$$x = x, \qquad y = kx, \qquad z = (A + Bk^2)x^2 + \cdots,$$

where x is the parameter. Now applying (5) we have

$$(6) \qquad c = \frac{A + Bk^2}{1 + k^2}.$$

Solving for k^2

$$(7) \qquad k^2 = \frac{A - c}{c - B},$$

and from this last expression we see that for real normal sections, that is for real values of k, c must lie between A and B. If then we measure on the normal $OA = A$ and $OB = B$ we see that the point C (the end of the normal curvature) will trace out the segment AB twice, since for each value of c there are two values of k. To the same value of c there correspond two values of k equal but opposite in sign, hence the directions in which the normal curvatures are equal are equally inclined to the lines of curvature.

If we denote the normal curvature in the direction of the lines of curvature, that is along the x- and y-axis, by c_1 and c_2 and their radii of curvature by R_1 and R_2 we have

$$c_1 = \frac{1}{R_1} = A, \qquad c_2 = \frac{1}{R_2} = B.$$

The mean curvature is then

$$H = \tfrac{1}{2}(A + B).$$

If we substitute H for c in (7) we find that the directions on the surface for which the normal curvature is H are $k = \pm 1$ and that therefore these directions bisect the angles between the lines of curvature and are perpendicular to each other. Now take two perpendicular directions k and $-(1/k)$ on the surface. For the normal curvature in these directions we have

$$c' = \frac{A + Bk^2}{1 + k^2}, \qquad c'' = \frac{Ak^2 + B}{1 + k^2},$$

[1] See Wilson's *Advanced Calculus*, p. 85.

and hence

$$\tfrac{1}{2}(c' + c'') = \tfrac{1}{2}(A + B) = H.$$

That is the average of the normal curvatures in two perpendicular directions is equal to the mean curvature. This means that C' and C'' are equally distant from the middle of the segment AB. If now we look on AB as a degenerate ellipse A and B are its vertices and H its center and to each direction on the surface corresponds a point on this degenerate ellipse. The ends of a diameter here will be points equally distant from the center but the values of k to which they correspond must be taken with opposite signs. With this convention then we can say that the ends of a diameter correspond to perpendicular directions. As appears then the two halves of the ellipse can be distinguished by saying that one side corresponds to positive values of k and the other side corresponds to negative values.

For a surface whose equation has the form (1), conjugate directions coincide with the directions of the conjugate diameters of the conic

$$Ax^2 + By^2 = 1.$$

Then the conjugate to any direction k is $-\ (A/Bk)$ and we find for the radii of curvature for normal sections in two such directions

$$R' = \frac{1 + k^2}{A + Bk^2}, \qquad R'' = \frac{A^2 + B^2k^2}{AB(A + Bk^2)}.$$

Hence

$$\tfrac{1}{2}(R' + R'') = \frac{A + B}{2AB} = \tfrac{1}{2}(R_1 + R_2) = R_m,$$

where R_m simply denotes the mean of the radii of curvature in the directions of the lines of curvature. The normal radii of curvature in two conjugate directions have the same relation to the mean radius of curvature that the curvature in two perpendicular directions has to the mean curvature. This property can then be used to define conjugate directions on the surface, that is two directions are conjugate if the mean of their normal radii of curvature is the mean radius of curvature.

The asymptotic directions are those in which the normal curvature is zero. These directions are $k = \pm\sqrt{-\ A/B}$. If then $A = 0$ both these directions coincide in the x-axis and likewise if $B = 0$ they both coincide in the y-axis. But if either $A = 0$ or $B = 0$ the segment AB has one end at O. If A and B have like signs the above values of k are imaginary and therefore the asymptotic directions are imaginary. This means that O is not a point of the segment AB. If A and B have different signs the asymptotic directions are real. In this case O is a point of the segment. A minimal surface is one for which H is zero for each point of the surface. Expressed in terms of the segment we see that for a minimal surface O must be the middle point of the segment.

The segment AB will then indicate about as much as the Dupin indicatrix.

The only exception being conjugate directions which coincide with conjugate diameters of the indicatrix. The segment shows that if O is not a point of AB the curvature of each normal section through O has the same sign and hence the surface is dome shaped at this point. This is usually called an elliptic point. If O is a point of the segment, not coincident with A or B, then some directions have positive normal curvature and some have negative. At such a point the surface is saddle shaped. These points are usually called hyperbolic. If O coincides with either A or B then in all directions the normal curvature has the same sign but there is a single value for which it is zero. These are called parabolic points.

The advantage of using the locus of the end of the normal curvature OC for an indicatrix shows up best in higher dimensions for the locus is, in general, not a degenerate ellipse. In higher dimensions there is a different direction of the normal to the surface for each direction in the tangent plane. The character of the surface is indicated by the position of the ellipse with reference to the surface point O. For example if the surface is a general one there is no relation between the plane of the ellipse and the point O. If the plane of the ellipse passes through O then, at this point, the surface has the character of a surface in a four-dimensional space. If O lies on the ellipse the point is called parabolic. If the ellipse degenerates into a segment, but the line on which it lies does not pass through O, the surface then has a special four-dimensional character having on it lines which have many of the properties of lines of curvature on surfaces in three dimensions. If the line of the segment passes through O then the surface has the character of a surface in ordinary space, at that point, and all the different cases which we have discussed in the preceding paragraph have the same meaning that they had in three dimensions. For the proof of these results for higher dimensions see "Differential Geometry of Surfaces in Hyperspace," by E. B. Wilson and C. L. E. Moore, *Proceedings of the American Academy of Arts and Sciences*, Vol. 52, 1916.

3. **Expression in Parametric Form.** The equation of the surface is often given in parametric form and therefore I shall indicate briefly how these same results would appear in parametric form. The formulas used can be found in any book on surface theory such as Eisenhart's *Differential Geometry.* Let the surface be given by the equations

$$x = \varphi(u, v), \qquad y = \chi(u, v), \qquad z = \psi(u, v). \tag{8}$$

If u and v are functions of a single paramenter or if v is a function of u then equations (8) are the equations of a curve traced on the surface. From differential geometry it is well known that the differential of arc of a curve traced on the surface is

$$ds^2 = Edu^2 + 2Fdu\,dv + Gdv^2,$$

where

$$E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2,$$

$$F = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}, \qquad G = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2.$$

If the parameter curves, that is the curves $u = \text{const.}$, $v = \text{const.}$, are orthogonal it is well known that $F = 0$. The curvature of any normal section of the surface is given by the formula

$$c = \frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2}, \tag{9}$$

where

$$L = \frac{1}{\sqrt{EG - F^2}} \begin{vmatrix} \dfrac{\partial^2 x}{\partial u^2} & \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{d^2 y}{\partial u^2} & \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \\[2ex] \dfrac{\partial^2 z}{\partial u^2} & \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{vmatrix}, \qquad M = \frac{1}{\sqrt{EG - F^2}} \begin{vmatrix} \dfrac{\partial^2 x}{\partial u \partial v} & \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial^2 y}{\partial u \partial v} & \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \\[2ex] \dfrac{\partial^2 z}{\partial u \partial v} & \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{vmatrix},$$

$$N = \frac{1}{\sqrt{EG - F^2}} \begin{vmatrix} \dfrac{\partial^2 x}{\partial v^2} & \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial^2 y}{\partial v^2} & \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \\[2ex] \dfrac{\partial^2 z}{\partial v^2} & \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{vmatrix}.$$

If the parameter curves are conjugate curves then $M = 0$. The directions of the lines of curvature are known to be conjugate and also orthogonal directions. Then if we take them for parameter curves both $M = 0$ and $F = 0$. The direction of a curve on the surface is defined by the ratio dv/du, that is, to each direction through a given point on the surface corresponds a single value of dv/du and vice versa. Then writing $k = dv/du$ and taking the lines of curvature for parameter lines equation (9) takes the simple form

$$(10)\quad c = \frac{L + Nk^2}{E + Gk^2}, \qquad \text{from which} \qquad (11)\quad k^2 = \frac{L - Ec}{cG - N}.$$

From the expressions for E and G we see that they are both positive and therefore from (11) for real values of k, that is for real directions on the surface, c must lie between L/E and N/G. Hence if we measure on the normal line a length equal to the normal curvature as was done in § 2 we see that the end of the curvature will trace out a segment from L/E to N/G. Since there are two values of k for a single value of c each point of this segment will be counted twice. The values of k corresponding to the same value of c differ only in sign, hence directions for which the normal curvatures are equal are equally inclined to the lines of curvature (the parameter lines in this case). We can see this by applying the formula for the angle between two directions k_1 and k_2

$$\cos\theta = \frac{E + Gk_1k_2}{\sqrt{E + Gk_1^2}\sqrt{E + Gk_2^2}}, \tag{12}$$

for the case in which the parameter curves are orthogonal. If we set $k_1 = k$, $k_2 = 0$ the above statement is verified.

If we denote the curvature of the lines of curvature by c_1 and c_2 and the corresponding radii of curvature by R_1, R_2 we have

$$c_1 = \frac{1}{R_1} = \frac{L}{E}, \qquad c_2 = \frac{1}{R_2} = \frac{N}{G}. \tag{13}$$

The mean curvature is then

$$H = \tfrac{1}{2}(c_1 + c_2) = \frac{LG + NE}{2EG}.$$

If H is substituted for c in (10) it is found that the two directions on the surface whose normal curvature is H are $k = \pm\sqrt{E/G}$. Substituting these values for k_1 and k_2 in (12) it is seen that these two directions are orthogonal. From (12) we see, in general, that two directions are orthogonal if

$$E + Gk_1k_2 = 0.$$

Now if we take a direction k and the direction perpendicular to it we find for the normal curvature in these two directions

$$c' = \frac{L + Nk^2}{E + Gk^2}, \qquad c'' = \frac{LG^2k^2 + E^2N}{EG(E + Gk^2)},$$

and hence

$$\tfrac{1}{2}(c' + c'') = \frac{LG + NE}{2EG} = H.$$

Hence the average of the curvatures in two perpendicular directions is constant and equal to the mean curvature H. We can now consider the segment as a degenerate ellipse just as in § 2.

From surface theory we also know that, if $M = 0$, conjugate directions are such that

$$L + Nk_1k_2 = 0. \tag{14}$$

For two such directions we have for the radii of curvature R', R''

$$R' = \frac{E + Gk^2}{L + Nk^2}, \qquad R'' = \frac{EN^2k^2 + GL^2}{NL(L + Nk^2)},$$

and hence

$$\tfrac{1}{2}(R' + R'') = \tfrac{1}{2}(R_1 + R_2) = R_m.$$

This last equation may then be used as a definition for conjugate directions. The remainder of the discussion would be the same as that given in § 2.